Simultaneous approximation estimates for Beta Szasz Mirakyan operators

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Abstract: In this paper, we abstract certain Beta-Szasz Mirakyan operators continuing the sequence of combined operators in simultaneous approximation and using Beta function of second kind. Simultaneous approximation theorem, Voronovskaya type asymptotic formula and error estimation for these operators are obtained here.
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1. Introduction

Srivastava-Gupta [6], Gupta-Lupas [3], Gupta-Noor [4] etc. proposed several types of combined operators. In this series we also propose a new sequence of summation integral type operators M_n named as Beta Szasz operators for $f \in C_\alpha[0,\infty)$, whereas

$$C_\alpha[0,\infty) = \{f \in C[0,\infty) : |f(t)| \leq Me^{\alpha t}\}$$

for some $M > 0$, $\alpha > 0$ and $x \in [0,\infty)$,

$$M_n(f,x) = \left(\frac{n}{n+1}\right) \sum_{v=1}^{\infty} b_{n,v}(x) \int_0^\infty q_{n,v}(t) f(t) dt, \tag{1.1}$$

where

$$\begin{aligned} q_{n,v}(x) &= e^{-nx}\frac{(nx)^v}{(v)!}, \\ b_{n,v}(x) &= \frac{1}{B(n+1,v)}\frac{x^{v-1}}{(1+x)^{n+v+1}} \end{aligned}$$

are Szasz basis function and Beta basis function respectively.

In this paper, we give exciting approximation theorems for the linear positive operators (1.1) such as simultaneous approximation, asymptotic formula and error estimation theorems . Many authors [1], [2], [5] etc. have discussed earlier these properties for several operators and found global results.

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2. Auxiliary Results

In this section we give some important lemmas related to the present operators.

Lemma 1. *For $m \in N^0$, the m^{th} order moment is obtained as*

$$U_{n,m}(x) = \frac{1}{n+1}\sum_{v=1}^{\infty} b_{n,v}(x)\left(\frac{v}{n+1} - x\right)^m,$$

where $U_{n,0}(x) = 1$, $U_{n,1}(x) = \frac{1+x}{n+1}$ and the recurrence formula is

$$(n+1)U_{n,m+1}(x)x(1+x)[U'_{n,m}(x) + mU_{n,m-1}(x)].$$

Consequently

$$U_{n,m}(x) = O(n^{-[m+1]/2}).$$

Lemma 2. *For some polynomial $q_{i,j,r}(x)$ independent of n and v, we have*

$$[x(1+x)]^r D^r(b_{n,v}(x)) = \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} (n+2)^i[(v-1)-(n+2)x]q_{i,j,r}(x)b_{n,v}(x)$$

where $D \equiv \frac{d}{dx}$.

The proof is too easy to prove.

Lemma 3. *We suppose that $T_{n,m}(x)$ represents the $m^{th}, m \geq 0$ central moment for the operators and is defined as*

$$T_{n,m}(x) = \left(\frac{n}{n+1}\right)\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty} q_{n,v}(t)(t-x)^m dt.$$

From here $T_{n,0}(x) = 1$, $T_{n,1}(x) = \frac{2(1+x)}{n}$, $T_{n,2}(x) = \frac{(n+6)(x+2)x+(n+6)+6}{n^2}$.

Consequently the recurrence formula of $T_{n,m}(x)$ for $m > 2$ is given by

$$\begin{aligned} nT_{n,m+1}(x) &= x(1+x)T'_{n,m}(x) + (m+2+2x)T_{n,m}(x) \\ &\quad + mx(x+2)T_{n,m-1}(x). \end{aligned}$$

Proof. The results of $T_{n,0}(x)$, $T_{n,1}(x)$ and $T_{n,2}(x)$ are obvious from the formula by substituting $m = 0, 1, 2$ respectively. To prove the recurrence formula firstly we give the two identities

$$\begin{aligned} x(1+x)b'_{n,v}(x) &= [(v-1)-(n+2)x]xb_{n,v}(x) \\ tq'_{n,v}(t) &= [v-nt]q_{n,v}(t). \end{aligned}$$

Now, we proceed as

$$T'_{n,m}(x) = \left(\frac{n}{n+1}\right)\sum_{v=1}^{\infty} b'_{n,v}(x)\int_0^{\infty} q_{n,v}(t)(t-x)^m dt - mT_{n,m-1}(x).$$

Using the above identities after the multiply of $x(1+x)$, we get

$$\begin{aligned}
& x(1+x)[T'_{n,m}(x) + mT_{n,m-1}(x)] \\
= & \left(\frac{n}{n+1}\right)\sum_{v=1}^{\infty} x(1+x)b'_{n,v}(x)\int_0^{\infty} q_{n,v}(t)(t-x)^m dt \\
= & \left(\frac{n}{n+1}\right)\sum_{v=1}^{\infty}[(v-1)-(n+2)x]b'_{n,v}(x)\int_0^{\infty} q_{n,v}(t)(t-x)^m dt \\
= & \left(\frac{n}{n+1}\right)\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty}[(v-nt)+n(t-x)-(1+2x)]q_{n,v}(t) \\
& \times(t-x)^m dt \\
= & \left(\frac{n}{n+1}\right)\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty} tq'_{n,v}(t)(t-x)^m dt + nT_{n,m+1}(x) \\
& -(1+2x)T_{n,m}(x) \\
= & \left(\frac{n}{n+1}\right)\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty} q'_{n,v}(t)(t-x)^{m+1} dt + \left(\frac{nx}{n+1}\right)\times \\
& \sum_{v=1}^{\infty} b_{n,v}(x)\int_0^{\infty} q'_{n,v}(t)(t-x)^m dt + nT_{n,m+1}(x) \\
& -(1+2x)T_{n,m}(x) \\
= & -(m+1)T_{n,m}(x) - mxT_{n,m-1}(x) + nT_{n,m+1}(x) \\
& -(1+2x)T_{n,m}(x) \\
= & +nT_{n,m+1}(x) - (m+2+2x)T_{n,m}(x) - mxT_{n,m-1}(x).
\end{aligned}$$

Rearranging both sides, we get the required. □

Lemma 4. *For given operators we can easily prove that*

$$M_n(t^r;x) = \frac{(n+r+1)!}{(n+1)!n^r}x^r + (r+1)r\frac{(n+r)!}{(n+1)!n^r}x^{r-1} + O(n^{-2}).$$

3. Main Results

In this section, we prove some important theorems.

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3.1. **Simultaneous approximation theorem.**

Theorem 1. *If $f \in C_\gamma[0,\infty), \gamma > 0$ such that f is r-times differentiable on $[0,\infty)$ then simultaneous approximation property for these operators is satisfied, that is*

$$\lim_{n\to\infty}[M_n^{(r)}(f,x) - f^{(r)}(x)] = 0.$$

Proof. Taylor's expansion of f is given by

$$f(t) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!}(t-x)^i + \epsilon(t,x)(t-x)^r,$$

where $\epsilon(t,x) \to 0$ as $t \to x$. Therefore taking

$$W_n(t,x) = \left(\frac{n}{n+1}\right)\sum_{v=1}^{\infty} b_{n,v}(x)q_{n,v}(t),$$

we have

$$\begin{aligned} M_n^{(r)}(f,x) &= \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(t,x)(t-x)^i dt \\ &\quad + \int_0^\infty W_n^{(r)}(t,x)\epsilon(t,x)(t-x)^r dt \\ &:= I_1 + I_2. \end{aligned}$$

To estimate I_1, we expand $(t-x)^i$ and then use Lemma 4 as

$$\begin{aligned} I_1 &= \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\sum_{k=0}^{i}\binom{i}{k}(-x)^{i-k}\int_0^\infty W_n^{(r)}(t,x)t^k dt \\ &= \frac{f^{(r)}(x)}{r!}\int_0^\infty W_n^{(r)}(t,x)t^r dt \\ &\to f^{(r)}(x) \end{aligned}$$

as $n\to\infty$. Now we consider about I_2. Using Lemma 2

$$\begin{aligned} |I_2| &= \left(\frac{n}{n+1}\right)\sum_{v=1}^{\infty}\sum_{\substack{2i+j\le r\\ i,j\ge 0}}(n+2)^i\frac{|q_{i,j,r}(x)|}{|x(1+x)|^r}|(v-1)-(n+2)x|^j \\ &\quad \times b_{n,v}(x)\int_0^\infty q_{n,v}(t,x)|\epsilon(t,x)||t-x|^r dt \\ &\le K_1\left(\frac{n}{n+1}\right)\sum_{v=1}^{\infty}\sum_{\substack{2i+j\le r\\ i,j\ge 0}}(n+2)^i\sum_{v=1}^{\infty}|(v-1)-(n+2)x|^j \end{aligned}$$

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$$\begin{aligned}
&\times b_{n,v}(x)\Big\{\epsilon\int_{|t-x|<\delta} q_{n,v}(t,x)|t-x|^r dt + K_2\int_{|t-x|\geq\delta} q_{n,v}(t,x)\\
&\times|t-x|^s dt\Big\}\\
:= \ & I_3 + I_4
\end{aligned}$$

where for a given $\epsilon > 0$ there exists a $\delta > 0$ such that $|\epsilon(t,x)| < \epsilon$ whenever $|t-x| < \delta$, and further we can find a constant K_2 such that $|\epsilon(t,x)||t-x|^r \leq K_2|t-x|^s$ for $|t-x| \geq \delta$ where $s \geq \{\gamma, r\}$, and

$$K_1 = \sup_{\substack{2i+j\leq r\\ i,j\geq 0}} \frac{|q_{i,j,r}(x)|}{|x(1+x)|^r}.$$

Using Lemma 1 and Lemma 3 after applying Schwarz inequality in I_3, we get

$$\begin{aligned}
I_3 \ &\leq \ K_3 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} (n+2)^i \left\{\frac{1}{n+1}\sum_{v=1}^{\infty} b_{n,v}(x)[(v-1)-(n+2)x]^{2j}\right\}^{1/2}\\
&\times\left\{n\int_0^\infty q_{n,v}(t,x)dt\right\}^{1/2}\left\{\frac{n}{n+1}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^\infty q_{n,v}(t,x)\right.\\
&\left.\times|t-x|^{2r}dt\right\}^{1/2}\\
&\leq \ \epsilon O(n^i).O(n^{j/2}).O(n^{-r/2}) = \epsilon O(1).
\end{aligned}$$

To compute I_4, we proceed in the similar manner as

$$\begin{aligned}
I_4 \ &\leq \ K_3 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} (n+2)^i \left\{\frac{1}{n+1}\sum_{v=1}^{\infty} b_{n,v}(x)[(v-1)-(n+2)x]^{2j}\right\}^{1/2}\\
&\times\left\{n\int_0^\infty q_{n,v}(t,x)dt\right\}^{1/2}\left\{\frac{n}{n+1}\sum_{v=1}^{\infty} b_{n,v}(x)\int_0^\infty q_{n,v}(t,x)\right.\\
&\left.\times|t-x|^{2s}dt\right\}^{1/2}\\
&\leq \ O(n^i).O(n^{j/2}).O(n^{-s/2})\\
&\leq \ O(n^{(r-s)/2} = o(1).
\end{aligned}$$

Thus for arbitrary $\epsilon > 0$, we get $I_2 = o(1)$. Together with the estimates of I_1 and I_2 we obtain the required proof of the theorem. □

3.2. Direct Theorem.

Theorem 2. *If $f \in C_\gamma[0,\infty), \gamma > 0$ and $r \leq m \leq (r+2)$. Again, if $f^{(m)}$ exists and is continuous on $(a-\eta, b+\eta)$, for n sufficiently large we have*

$$\|M_n^{(r)}(f,x)-f^{(r)}(x)\| \leq K_4 n^{-1}\sum_{i=r}^{m}\|f^{(i)}\|+K_5\omega(f^{(r+1)},n^{-1/2})+O(n^{-2}),$$

where k_4 and K_5 are constants independent of n v. $\omega(f,\delta)$ s the modulus of continuity of f on $(a-\eta,b+\eta)$ and $\|.\|$ represents the sup-norm on the interval $[0,\infty)$.

Proof. Taylor series expansion of f is given by

$$\begin{aligned} f(t) &= \sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}(t-x)^i+(t-x)^m\chi(t)\frac{f^{(m)}(\xi)-f^{(m)}(x)}{m!} \\ &\quad +(1-\chi(t))h(t,x), \end{aligned}$$

where $t<\xi<x$ and $\chi(t)$ is the characteristic function on $(a-\eta,b+\eta)$. Further we have for $t\in(a-\eta,b+\eta)$ and $x\in[a,b]$

$$f(t)=\sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}(t-x)^i+(t-x)^m\frac{f^{(m)}(\xi)-f^{(m)}(x)}{m!}$$

and for $t\in[0,\infty)\backslash(a-\eta,b+\eta)$ we define

$$h(t,x)=f(t)-\sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}(t-x)^i.$$

Then

$$\begin{aligned} & M_n^{(r)}(f,x)-f^{(r)}(x) \\ = & \left\{\sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(t,x)(t-x)^i dt - f^{(r)}(x)\right\} \\ & +\left\{\int_0^\infty W_n^{(r)}(t,x)\frac{f^{(m)}(\xi)-f^{(m)}(x)}{m!}(t-x)^m\chi(t)dt\right\} \\ & +\left\{\int_0^\infty W_n^{(r)}(t,x)(1-\chi(t))h(t,x)dt\right\} \\ := & J_1+J_2+J_3. \end{aligned}$$

We use Lemma 4 to estimate J_1, as below

$$J_1 = \sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(t,x)(t-x)^i dt - f^{(r)}(x)$$

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$$
\begin{aligned}
&= \sum_{i=0}^{m} \frac{f^{(i)}(x)}{i!} \sum_{k=0}^{i} \binom{i}{k} (-x)^{i-k} \int_0^\infty W_n^{(r)}(t,x) t^k dt - f^{(r)}(x) \\
&= \sum_{i=0}^{m} \frac{f^{(i)}(x)}{i!} \sum_{k=0}^{i} \binom{i}{k} (-x)^{i-k} \frac{\partial^r}{\partial x^r} \Bigg[\frac{(n+k+1)!}{(n+1)! n^k} x^k + (k+1)k \\
&\quad \times \frac{(n+k)!}{(n+1)! n^k} x^{k-1} + O(n^{-2}) \Bigg] - f^{(r)}(x).
\end{aligned}
$$

Therefore we can say that

$$\|J_1\| \leq K_4 n^{-1} \sum_{i=r}^{m} \|f^{(i)}\| - f^{(r)}(x)$$

uniformly in $x \in [a,b]$. Now we proceed for J_2.

$$
\begin{aligned}
|J_2| &\leq \int_0^\infty |W_n^{(r)}(t,x)| \frac{|f^{(m)}(\xi) - f^{(m)}(x)|}{m!} |t-x|^m \chi(t) dt \\
&\leq \frac{\omega(f^{(m)}, \delta)}{m!} \int_0^\infty |W_n^{(r)}(t,x)| \left(1 - \frac{|t-x|}{\delta}\right) |t-x|^m dt \\
&\leq \frac{\omega(f^{(m)}, \delta)}{m!} \Bigg[\int_0^\infty |W_n^{(r)}(t,x)| |t-x|^m dt + \int_0^\infty |W_n^{(r)}(t,x)| \\
&\quad \times |t-x|^{m+1} \delta^{-1} dt \Bigg].
\end{aligned}
$$

As in the previous theorem, for some $s > 0$ and $\delta = -1/2$ we get

$$
\begin{aligned}
\|J_2\| &\leq \frac{\omega(f^{(m)}, \delta)}{m!} [O(n^{(r-m)/2}) + n^{1/2} O(n^{(r-m-1)/2}) + O(n^{-s})] \\
&\leq K_5 \omega(f^{(m)}, \delta) . O(n^{-(m-r)/2}).
\end{aligned}
$$

For J_3, since $t \in [0,\infty) \backslash (a-\eta, b+\eta)$ so we can choose a $\delta > 0$ in such a way that $|t-x| \geq \delta$ for all $x \in [a,b]$. Applying Lemma 2 and then from Theorem 1

$$
\begin{aligned}
|J_3| &= \left(\frac{n}{n+1}\right) \sum_{\substack{2i+j \leq r \\ i,j \geq 0}} (n+2)^i \frac{|q_{i,j,r}(x)|}{|x(1+x)|^r} \sum_{v=1}^{\infty} |(v-1) - (n+2)x|^j \\
&\quad \times b_{n,v}(x) \int_{|t-x| \geq \delta} q_{n,v}(t,x) |h(t,x)| dt
\end{aligned}
$$

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$$\leq K_1 \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} (n+2)^i \Bigg\{ \left(\frac{1}{n+1}\right) \sum_{v=1}^{\infty} b_{n,v}(x)|(v-1)- (n+2)x|^{2j} \Bigg\}^{1/2} \Bigg\{ n\int_0^{\infty} q_{n,v}(t,x)dt \Bigg\}^{1/2} \Bigg\{ \left(\frac{n}{n+1}\right) \sum_{v=1}^{\infty} b_{n,v}(x) \times \int_0^{\infty} q_{n,v}(t,x)|h(t,x)|dt \Bigg\}^{1/2}.$$

Hence from Lemma 1 and Lemma 3

$$\|J_3\| \leq K_1.O(n^i).O(n^j).O(n^{-\beta}),$$

where $\beta \geq \{\gamma, m\}$ is an integer for which there exists a constant K_6 such that $|h(t,x)| \leq K_6|t-x|^{\beta}$ for $|t-x| \geq \delta$ Thus $\|J_3\| = O(n^{-q})$ for some $q > 0$ uniformly on $[a,b]$. Gathering J_1, J_2 and J_3, we get the result. □

3.3. **Asymptotic Formula.**

Theorem 3. *If* $f \in C_{\gamma}[0,\infty), \gamma > 0$ *such that* $f^{(r+2)}$ *exists at* $x \in [0,\infty)$ *then*

$$\lim_{n\to\infty} n[M_n^{(r)}(f,x) - f^{(r)}(x)] = \frac{r(r+3)}{2} f^{(r)}(x) + (r+2)(1+x) \times f^{(r+1)}(x) + \frac{x(x+1)}{2} f^{(r+2)}(x).$$

Proof. Taylor's expansion of f is given by

$$f(t) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \epsilon(t,x)(t-x)^{r+2},$$

where $\epsilon(t,x) \to 0$ as $t \to x$. Therefore taking as in Theorem 1

$$\begin{aligned} n[M_n^{(r)}(f,x) - f^{(r)}] &= n\Bigg\{ \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \int_0^{\infty} W_n^{(r)}(t,x)(t-x)^i dt - f^{(r)}(x) \Bigg\} + \int_0^{\infty} W_n^{(r)}(t,x)\epsilon(t,x)(t-x)^{r+2} dt \\ &:= E_1 + E_2. \end{aligned}$$

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To find E_1, we use Lemma 3 and get

$$\begin{aligned}
E_1 &= n\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\sum_{k=0}^{i}\left(\begin{array}{c} i \\ k \end{array}\right)(-x)^{i-k}\int_0^\infty W_n^{(r)}(t,x)t^k dt - nf^{(r)}(x) \\
&= n\frac{f^{(r)}(x)}{r!}\left\{M_n^{(r)}(t^r,x)-r!\right\} + n\frac{f^{(r+1)}(x)}{(r+1)!}\Big\{M_n^{(r)}(t^{r+1},x)+ \\
&\quad (r+1)(-x)M_n^{(r)}(t^r,x)\Big\} + n\frac{f^{(r+2)}(x)}{(r+2)!}\Big\{M_n^{(r)}(t^{r+2},x)+(r+2) \\
&\quad \times(-x)M_n^{(r)}(t^{r+1},x)+\frac{(r+1)(r+2)}{2!}x^2M_n^{(r)}(t^r,x)\Big\}.
\end{aligned}$$

For each $x\in[0,\infty)$, applying Lemma 4

$$\begin{aligned}
E_1 &= nf^{(r)}(x)\left[\frac{(n+r+1)!}{(n+1)!n^r}-1\right] + n\frac{f^{(r+1)}(x)}{(r+1)!}\Bigg[\Bigg\{\frac{(n+r+2)!}{(n+1)!n^{r+1}} \\
&\quad \times(r+1)!x+(r+2)(r+1)\frac{(n+r+1)!}{(n+1)!n^{r+1}}r!\Bigg\}+(r+1)(-x) \\
&\quad \times\frac{(n+r+1)!}{(n+1)!n^r}r!\Bigg] + n\frac{f^{(r+2)}(x)}{(r+2)!}\Bigg[\Bigg\{\frac{(n+r+3)!}{(n+1)!n^{r+2}}\frac{(r+2)!}{2}x^2 \\
&\quad +(r+1)(r+2)\frac{(n+r+2)!}{(n+1)!n^{r+2}}(r+1)!x\Bigg\}+(r+2)(-x) \\
&\quad \times\Bigg\{\frac{(n+r+2)!}{(n+1)!n^{r+1}}(r+1)!x+(r+2)(r+1)\frac{(n+r+1)!}{(n+1)!n^{r+1}}r!\Bigg\} \\
&\quad +\frac{(r+1)(r+2)}{2!}x^2\frac{(n+r+1)!}{(n+1)!n^r}r!\Bigg] + O(n^{-2}).
\end{aligned}$$

Taking limit as $n\to\infty$ on right side, the coefficients of $f^{(r)}(x)$, $f^{(r+1)}(x)$ and $f^{(r+2)}(x)$ are $\frac{r(r+3)}{2}$, $(r+2)(1+x)$ and $\frac{x(x+1)}{2}$ respectively. In order to complete the theorem we can easily show that $J_2\to 0$ as $n\to\infty$ accordingly as in Theorem 1. $\square$

Remark 1. *In particular the asymptotic formula in ordinary approximation for bounded functions can easily be found as*

$$\lim_{n\to\infty} n[M_n^{(r)}(f,x)-f^{(r)}(x)] = 2(1+x)f^{(1)}(x)+\frac{x(x+1)}{2}f^{(2)}(x).$$

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